J Stat Phys (2010) 138: 598–618
DOI 10.1007/s10955-009-9876-0

Localization for Linear Stochastic Evolutions

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Received: 25 June 2009 / Accepted: 10 November 2009 / Published online: 3 December 2009

Abstract We consider a discrete-time stochastic growth model on the d-dimensional lattice with non-negative real numbers as possible values per site. The growth model describes various interesting examples such as oriented site/bond percolation, directed polymers in random environment, time discretizations of the binary contact path process. We show the equivalence between the slow population growth and a localization property in terms of "replica overlap". The main novelty of this paper is that we obtain this equivalence even for models with positive probability of extinction at finite time. In the course of the proof, we characterize, in a general setting, the event on which an exponential martingale vanishes in the limit.

Keywords Linear stochastic evolutions · Localization · Slow growth phase

1 Introduction

We write $\mathbb{N}=\{0,1,2,\ldots\}$, $\mathbb{N}^*=\{1,2,\ldots\}$ and $\mathbb{Z}=\{\pm x; x\in\mathbb{N}\}$. For $x=(x_1,\ldots,x_d)\in\mathbb{R}^d$, $|x|$ stands for the ℓ^1-norm: $|x|=\sum_{i=1}^d |x_i|$. For $\xi=(\xi_x)_{x\in\mathbb{Z}^d}\in\mathbb{R}^{\mathbb{Z}^d}$, $|\xi|=\sum_{x\in\mathbb{Z}^d}|\xi_x|$. Let $(\Omega,\mathcal{F},P)$ be a probability space. We write $P[X]=\int X\,dP$ and $P[X:A]=\int_A X\,dP$ for a random variable X and an event A. For events $A,B\subset\Omega$, $A\subset B$ a.s. means that $P(A\backslash B)=0$. Similarly, $A=B$ a.s. means that $P(A\backslash B)=P(B\backslash A)=0$.

1.1 The Oriented Site Percolation (OSP)

We start by discussing the *oriented site percolation* as a motivating example. Let $\eta_{t,y}$, $(t,y)\in\mathbb{N}^*\times\mathbb{Z}^d$ be $\{0,1\}$-valued i.i.d. random variables with $P(\eta_{t,y}=1)=p\in(0,1)$. The site (t,y) with $\eta_{t,y}=1$ and $\eta_{t,y}=0$ are referred to respectively as *open* and *closed*. An *open*

The author was supported in part by JSPS Grant-in-Aid for Scientific Research, Kiban (C) 17540112.

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oriented path from $(0,0)$ to $(t,y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s,x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$, $|x_s - x_{s-1}| = 1$, $\eta_{s,x_s} = 1$ for all $s = 1, \dots, t$. For oriented percolation, it is traditional to discuss the presence/absence of the open oriented paths to certain time-space location. On the other hand, the model exhibits another type of phase transition, if we look at not only the presence/absence of the open oriented paths, but also their number. Let $N_{t,y}$ be the number of open oriented paths from $(0,0)$ to (t,y) and let $|N_t| = \sum_{y \in \mathbb{Z}^d} N_{t,y}$ be the total number of open oriented paths from $(0,0)$ to the "level" t. If we regard each open oriented path $\{(s,x_s)\}_{s=0}^t$ as a trajectory of a particle, then $N_{t,y}$ is the number of the particles which occupy the site y at time t.

We now note that $|\overline{N}_t| \stackrel{\text{def.}}{=} (2dp)^{-t}|N_t|$ is a martingale, since each open oriented path from $(0,0)$ to (t,y) branches and survives to the next level via $2d$ neighbors of y, each of which is open with probability p. Thus, by the martingale convergence theorem, the following limit exists a.s.:

$$|\overline{N}_\infty| \stackrel{\text{def}}{=} \lim_{t \to \infty} |\overline{N}_t|.$$

Moreover,

(i) If $d \geq 3$ and p is large enough, then, $P(|\overline{N}_\infty| > 0) > 0$, which means that, at least with positive probability, the total number of paths $|N_t|$ is of the same order as its expectation $(2pd)^t$ as $t \to \infty$.
(ii) If $d = 1, 2$, then for all $p \in (0,1)$, $P(|\overline{N}_\infty| = 0) = 1$, which means that the total number of paths $|N_t|$ is of smaller order than its expectation $(2pd)^t$ a.s. as $t \to \infty$. Moreover, there is a non-random constant $c > 0$ such that $|\overline{N}_t| = \mathcal{O}(\exp(-ct))$ a.s. as $t \to \infty$.

This phase transition was predicted by T. Shiga in late 1990's and the proof was given recently in [1, 18].

We denote the density of the population by:

$$\rho_t(x) = \frac{N_{t,x}}{|N_t|} \mathbf{1}_{\{|N_t|>0\}}, \quad t \in \mathbb{N},\ x \in \mathbb{Z}^d. \tag{1.1}$$

Here and in what follows, we adopt the following convention. For a random variable X defined on an event A, we define the random variable $X\mathbf{1}_A$ by $X\mathbf{1}_A = X$ on A and $X\mathbf{1}_A = 0$ outside A. Interesting objects related to the density would be

$$\rho_t^* = \max_{x \in \mathbb{Z}^d} \rho_t(x) \quad \text{and} \quad \mathcal{R}_t = |\rho_t^2| = \sum_{x \in \mathbb{Z}^d} \rho_t(x)^2. \tag{1.2}$$

ρ_t^* is the density at the most populated site, while $\mathcal{R}_t$ is the probability that two particles picked up randomly from the total population at time t are at the same site. We call $\mathcal{R}_t$ the *replica overlap*, in analogy with the spin glass theory. Clearly, $(\rho_t^*)^2 \leq \mathcal{R}_t \leq \rho_t^*$. These quantities convey information on localization/delocalization of the particles. Roughly speaking, large values of ρ_t^* or $\mathcal{R}_t$ indicate that most of the particles are concentrated on small numbers of "favorite sites" (*localization*), whereas small values of them imply that the particles are spread out over large number of sites (*delocalization*).

As applications of results in this paper, we get the following result. It says that, in the presence of an infinite open path, the slow growth $|\overline{N}_\infty| = 0$ is equivalent to a localization property $\overline{\lim}_{t \to \infty} \mathcal{R}_t \geq c > 0$. Here, and in what follows, a *constant* always means a *non-random constant*.

Theorem 1.1.1

(a) *If* $P(|\overline{N}_\infty|>0)>0$, *then* $\sum_{t\ge 1}\mathcal{R}_t<\infty$ *a.s.*
(b) *If* $P(|\overline{N}_\infty|=0)=1$, *then there exists a constant* $c>0$ *such that*:

$$\{|N_t|>0 \text{ for all } t\in\mathbb{N}\}=\left\{\overline{\lim_{t\to\infty}}\mathcal{R}_t\ge c\right\}\quad a.s. \tag{1.3}$$

Note that $P(|\overline{N}_\infty|=0)=1$ for all $p\in(0,1)$ if $d\le 2$. Thus, (1.3) in particular means that, if $d\le 2$, the path localization $\overline{\lim}_{t\to\infty}\mathcal{R}_t\ge c$ occurs a.s. on the event of percolation. Theorem 1.1.1 is shown at the end of Sect. 1.4 as a consequence of more general results for linear stochastic evolutions.

1.2 The Linear Stochastic Evolution

We now introduce the framework of this article. Let $A_t=(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$, $t\in\mathbb{N}^*$ be a sequence of random matrices on a probability space $(\Omega,\mathcal{F},P)$ such that:

$$A_1, A_2, \dots \text{ are i.i.d.} \tag{1.4}$$

Here are the set of assumptions we assume for A_1:

$$A_1 \text{ is not a constant matrix.} \tag{1.5}$$

$$A_{1,x,y}\ge 0 \quad \text{for all } x,y\in\mathbb{Z}^d. \tag{1.6}$$

$$\text{The columns } \{A_{1,\cdot,y}\}_{y\in\mathbb{Z}^d} \text{ are independent.} \tag{1.7}$$

$$P[A_{1,x,y}^3]<\infty \quad \text{for all } x,y\in\mathbb{Z}^d. \tag{1.8}$$

$$A_{1,x,y}=0 \quad \text{a.s. if } |x-y|>r_A \text{ for some non-random } r_A\in\mathbb{N}. \tag{1.9}$$

$$(A_{1,x+z,y+z})_{x,y\in\mathbb{Z}^d}\stackrel{\text{law}}{=}A_1 \quad \text{for all } z\in\mathbb{Z}^d. \tag{1.10}$$

$$\text{The set } \left\{x\in\mathbb{Z}^d;\ \sum_{y\in\mathbb{Z}^d}a_{x+y}a_y\ne 0\right\} \text{ contains a linear basis of } \mathbb{R}^d,$$
$$\text{where } a_y=P[A_{1,0,y}]. \tag{1.11}$$

Depending on the results we prove in the sequel, some of these conditions can be relaxed. However, we choose not to bother ourselves with the pursuit of the minimum assumptions for each result.

We define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $[0,\infty)^{\mathbb{Z}^d}$ by:

$$\sum_{x\in\mathbb{Z}^d}N_{t-1,x}A_{t,x,y}=N_{t,y},\quad t\in\mathbb{N}^*. \tag{1.12}$$

In this article, we suppose that the initial state N_0 is given by "a single particle at the origin":

$$N_0=\left(\delta_{0,x}\right)_{x\in\mathbb{Z}^d}. \tag{1.13}$$

Here and in what follows, $\delta_{x,y}=\mathbf{1}_{\{x=y\}}$ for $x,y\in\mathbb{Z}^d$. If we regard $N_t\in[0,\infty)^{\mathbb{Z}^d}$ as a row vector, (1.12) can be interpreted as:

$$N_t=N_0A_1A_2\cdots A_t,\quad t=1,2,\dots.$$

The Markov chain defined above can be thought of as the time discretization of the linear particle system considered in the last chapter in T. Liggett's book [11, Chap. IX]. Thanks to the time discretization, the definition is considerably simpler here. Though we *do not* assume in general that $(N_t)_{t\in\mathbb{N}}$ takes values in $\mathbb{N}^{\mathbb{Z}^d}$, we refer $N_{t,y}$ as the "number of particles" at time-space (t, y), and $|N_t|$ as the "total number of particles" at time t.

We now see that various interesting examples are included in this framework. We recall the notation a_y from (1.11).

- *Generalized oriented site percolation* (*GOSP*): We generalize OSP as follows. Let $\eta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be $\{0, 1\}$-valued i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in [0, 1]$ and let $\zeta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0, 1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0, 1]$, which are independent of $\eta_{t,y}$'s. To exclude trivialities, we assume that either p or q is in $(0, 1)$. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.12) with:

$$A_{t,x,y} = \mathbf{1}_{|x-y|=1}\eta_{t,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented site percolation* (GOSP). Thus, the OSP is the special case $(q = 0)$ of GOSP. The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$\begin{aligned} a_y &= p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \\ P[A_{t,x,y}A_{t,\tilde{x},y}] &= \begin{cases} q & \text{if } x = \tilde{x} = y, \\ p & \text{if } |x-y| = |\tilde{x}-y| = 1, \\ a_{y-x}a_{y-\tilde{x}} & \text{if otherwise.} \end{cases} \end{aligned} \tag{1.14}$$

In particular, we have $|a| = 2dp + q$ (recall that $|a| = \sum_y a_y$).

- *Generalized oriented bond percolation* (*GOBP*): Let $\eta_{t,x,y}$, $(t, x, y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$ be $\{0, 1\}$-valued i.i.d.random variables with $P(\eta_{t,x,y} = 1) = p \in [0, 1]$ and let $\zeta_{t,y}$, $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ be another $\{0, 1\}$-valued i.i.d. random variables with $P(\zeta_{t,y} = 1) = q \in [0, 1]$, which are independent of $\eta_{t,y}$'s. We refer to the process $(N_t)_{t\in\mathbb{N}}$ defined by (1.12) with:

$$A_{t,x,y} = \mathbf{1}_{\{|x-y|=1\}}\eta_{t,x,y} + \delta_{x,y}\zeta_{t,y}$$

as the *generalized oriented bond percolation* (GOBP). We call the special case $q = 0$ *oriented bond percolation* (OBP). To interpret the definition, let us call the pair of time-space points $\langle(t-1, x), (t, y)\rangle$ a *bond* if $|x - y| \le 1$, $(t, x, y) \in \mathbb{N}^* \times \mathbb{Z}^d \times \mathbb{Z}^d$. A bond $\langle(t-1, x), (t, y)\rangle$ with $|x - y| = 1$ is said to be *open* if $\eta_{t,x,y} = 1$, and a bond $\langle(t-1, y), (t, y)\rangle$ is said to be *open* if $\zeta_{t,y} = 1$. For GOBP, an *open oriented path* from $(0, 0)$ to $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is a sequence $\{(s, x_s)\}_{s=0}^t$ in $\mathbb{N} \times \mathbb{Z}^d$ such that $x_0 = 0$, $x_t = y$ and bonds $\langle(s-1, x_{s-1}), (s, x_s)\rangle$ are open for all $s = 1, \dots, t$. If $N_0 = (\delta_{0,y})_{y\in\mathbb{Z}^d}$, then, the number of open oriented paths from $(0, 0)$ to $(t, y) \in \mathbb{N}^* \times \mathbb{Z}^d$ is given by $N_{t,y}$.

  The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = p\mathbf{1}_{\{|y|=1\}} + q\delta_{y,0}, \qquad P[A_{t,x,y}A_{t,\tilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \tilde{x}, \\ a_{y-x}a_{y-\tilde{x}} & \text{if otherwise.} \end{cases} \tag{1.15}$$

In particular, we have $|a| = 2dp + q$.

- *Directed polymers in random environment* (*DPRE*): Let $\{\eta_{t,y}; (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$ be i.i.d. with $\exp(\lambda(\beta)) \stackrel{\text{def}}{=} P[\exp(\beta\eta_{t,y})] < \infty$ for any $\beta \in (0, \infty)$. The following expectation is called the partition function of the *directed polymers in random environment*:

$$N_{t,y} = P_S^0\left[\exp\left(\beta\sum_{u=1}^{t}\eta_{u,S_u}\right) : S_t = y\right], \quad (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d,$$

where $((S_t)_{t\in\mathbb{N}}, P_S^x)$ is the simple random walk on $\mathbb{Z}^d$. We refer the reader to a review paper [6] and the references therein for more information. Starting from $N_0 = (\delta_{0,x})_{x\in\mathbb{Z}^d}$, the above expectation can be obtained inductively by (1.12) with:

$$A_{t,x,y} = \frac{\mathbf{1}_{|x-y|=1}}{2d}\exp(\beta\eta_{t,y}).$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{e^{\lambda(\beta)}\mathbf{1}_{\{|y|=1\}}}{2d}, \qquad P[A_{t,x,y}A_{t,\widetilde{x},y}] = e^{\lambda(2\beta)-2\lambda(\beta)}a_{y-x}a_{y-\widetilde{x}}. \tag{1.16}$$

In particular, we have $|a| = e^{\lambda(\beta)}$.

- *The binary contact path process* (*BCPP*): The binary contact path process is a continuous-time Markov process with values in $\mathbb{N}^{\mathbb{Z}^d}$, originally introduced by D. Griffeath [9]. In this article, we consider a discrete-time variant as follows. Let

$$\{\eta_{t,y} = 0, 1; (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\}, \quad \{\zeta_{t,y} = 0, 1; (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\},$$
$$\{e_{t,y}; (t, y) \in \mathbb{N}^* \times \mathbb{Z}^d\}$$

be families of i.i.d. random variables with $P(\eta_{t,y} = 1) = p \in (0, 1]$, $P(\zeta_{t,y} = 1) = q \in [0, 1]$, and $P(e_{t,y} = e) = \frac{1}{2d}$ for each $e \in \mathbb{Z}^d$ with $|e| = 1$. We suppose that these three families are independent of each other. Starting from an $N_0 \in \mathbb{N}^{\mathbb{Z}^d}$, we define a Markov chain $(N_t)_{t\in\mathbb{N}}$ with values in $\mathbb{N}^{\mathbb{Z}^d}$ by:

$$N_{t+1,y} = \eta_{t+1,y}N_{t,y-e_{t+1,y}} + \zeta_{t+1,y}N_{t,y}, \quad t \in \mathbb{N}.$$

We interpret the process as the spread of an infection, with $N_{t,y}$ infected individuals at time t at the site y. The $\zeta_{t+1,y}N_{t,y}$ term above means that these individuals remain infected at time $t+1$ with probability q, and they recover with probability $1-q$. On the other hand, the $\eta_{t+1,y}N_{t,y-e_{t+1,y}}$ term means that, with probability p, a neighboring site $y - e_{t+1,y}$ is picked at random (say, the wind blows from that direction), and $N_{t,y-e_{t+1,y}}$ individuals at site y are infected anew at time $t + 1$. This Markov chain is obtained by (1.12) with:

$$A_{t,x,y} = \eta_{t,y}\mathbf{1}_{e_{t,y}=y-x} + \zeta_{t,y}\delta_{x,y}.$$

The covariances of $(A_{t,x,y})_{x,y\in\mathbb{Z}^d}$ can be seen from:

$$a_y = \frac{p\mathbf{1}_{\{|y|=1\}}}{2d} + q\delta_{0,y}, \qquad P[A_{t,x,y}A_{t,\widetilde{x},y}] = \begin{cases} a_{y-x} & \text{if } x = \widetilde{x}, \\ \delta_{x,y}qa_{y-\widetilde{x}} + \delta_{\widetilde{x},y}qa_{y-x} & \text{if } x \neq \widetilde{x}. \end{cases} \tag{1.17}$$

In particular, we have $|a| = p + q$.

Remark The branching random walk in random environment considered in [10, 15–17] can also be considered as a "close relative" to the models considered here, although it does not exactly fall into our framework.

1.3 The Regular and Slow Growth Phases

We now recall the following facts and notion from [18, Lemmas 1.3.1 and 1.3.2]. Let $\mathcal{F}_t$ be the σ-field generated by $A_1, \dots, A_t$.

Lemma 1.3.1 *Define* $\overline{N}_t = (\overline{N}_{t,x})_{x \in \mathbb{Z}^d}$ *by*:

$$\overline{N}_{t,x} = |a|^{-t} N_{t,x}. \tag{1.18}$$

(a) $(|\overline{N}_t|, \mathcal{F}_t)_{t \in \mathbb{N}}$ *is a martingale, and therefore, the following limit exists a.s.*

$$|\overline{N}_\infty| = \lim_{t \to \infty} |\overline{N}_t|. \tag{1.19}$$

(b) *Either*

$$P[|\overline{N}_\infty|] = 1 \text{ or } 0. \tag{1.20}$$

Moreover, $P[|\overline{N}_\infty|] = 1$ *if and only if the limit* (1.19) *is convergent in* $\mathbb{L}^1(P)$.

We will refer to the former case of (1.20) as *regular growth phase* and the latter as *slow growth phase*.

The regular growth means that, at least with positive probability, the growth of the "total number" $|N_t|$ of particles is of the same order as its expectation $|a|^t |N_0|$. On the other hand, the slow growth means that, almost surely, the growth of $|N_t|$ is slower than its expectation.

We now recall from [1] and [18, Theorems 3.1.1 and 3.2.1] the following criterion for slow growth phase.

Proposition 1.3.2 $P(|\overline{N}_\infty| = 0) = 1$ *if* $d = 1, 2$, *or if*:

$$\sum_{y \in \mathbb{Z}^d} P\left[A_{1,0,y} \ln A_{1,0,y}\right] > |a| \ln |a|. \tag{1.21}$$

The condition (1.21) roughly says that the matrix A_1 is "random enough". For DPRE, (1.21) is equivalent to $\beta\lambda'(\beta) - \lambda(\beta) > \ln(2d)$.

1.4 The Results

We introduce the following additional condition, which says that the entries of the matrix A_1 are positively correlated in the following weak sense: there is a constant $\gamma \in (1, \infty)$ such that:

$$\sum_{x, \widetilde{x}, y \in \mathbb{Z}^d} \left(P[A_{1,x,y} A_{1,\widetilde{x},y}] - \gamma a_{y-x} a_{y-\widetilde{x}}\right) \xi_x \xi_{\widetilde{x}} \geq 0 \tag{1.22}$$

for all $\xi \in [0, \infty)^{\mathbb{Z}^d}$ such that $|\xi| < \infty$.

Remark Clearly, (1.22) is satisfied if there is a constant $\gamma \in (1, \infty)$ such that:

$$P[A_{1,x,y}, A_{1,\widetilde{x},y}] \geq \gamma a_{y-x} a_{y-\widetilde{x}} \quad \text{for all } x, \widetilde{x}, y \in \mathbb{Z}^d. \tag{1.23}$$

For OSP and DPRE, we see from (1.14) and (1.16) that (1.23) holds with:

$$\gamma = 1/p \quad \text{and} \quad \exp(\lambda(2\beta) - 2\lambda(\beta))$$

respectively for OSP and DPRE. For GOSP, GOBP and BCPP, (1.23) is no longer true. However, one can check (1.22) for them with:

$$\gamma = 1 + \begin{cases} \frac{2dp(1-p)+q(1-q)}{(2dp+q)^2} & \text{for GOSP and GOBP,} \\ \frac{p(1-p)+q(1-q)}{(p+q)^2} & \text{for BCPP} \end{cases}$$

(see [18, remarks after Theorem 3.2.1]).

We define the density $\rho_t(x)$ and the replica overlap $\mathcal{R}_t$ in the same way as (1.1) and (1.2).

We first show that, on the event of survival, the slow growth is equivalent to the localization:

Theorem 1.4.1 *Suppose* (1.22).

(a) *If* $P(|\overline{N}_\infty| > 0) > 0$, *then* $\sum_{t\ge 0} \mathcal{R}_t < \infty$ *a.s.*

(b) *If* $P(|\overline{N}_\infty| = 0) = 1$, *then*

$$\{\text{survival}\} = \left\{ \sum_{t\ge 0} \mathcal{R}_t = \infty \right\} \quad a.s. \tag{1.24}$$

where $\{\text{survival}\} \stackrel{\text{def}}{=} \{|N_t| > 0 \text{ for all } t \in \mathbb{N}\}$. *Moreover, there exists a constant* $c > 0$ *such that almost surely,*

$$|\overline{N}_t| \le \exp\left(-c \sum_{1\le s\le t-1} \mathcal{R}_s \right) \quad \textit{for all large enough } t\textit{'s}. \tag{1.25}$$

Remark As can be seen from the proof (cf. Proposition 2.1.1(a)), (1.24) is true even without assuming (1.22) and with (1.8) replaced by a weaker assumption:

$$P[A_{1,x,y}^2] < \infty \quad \text{for all } x, y \in \mathbb{Z}^d. \tag{1.26}$$

Theorem 1.4.1 says that, conditionally on survival, the slow growth $|\overline{N}_\infty| = 0$ is equivalent to the localization $\sum_{t\ge 0} \mathcal{R}_t = \infty$. We emphasize that this is the first case in which a result of this type is obtained for models with positive probability of extinction at finite time (i.e., $P(|N_t| = 0) > 0$ for finite t). Similar results have been known before only in the case where no extinction at finite time is allowed, i.e., $|N_t| > 0$ for all $t \ge 0$, e.g., [4, Theorem 1.1], [5, Theorem 1.1], [7, Theorem 2.3.2], [10, Theorem 1.3.1]. The argument in the previous literature is roughly to show that

$$-\ln |\overline{N}_t| \asymp \sum_{u=0}^{t-1} \mathcal{R}_u \quad \text{a.s. as } t \to \infty \tag{1.27}$$

by using Doob's decomposition of the supermartingale $\ln |\overline{N}_t|$ ("$\asymp$" above means the asymptotic upper and lower bounds with positive multiplicative constants). This argument does not seem to be directly transportable to the case where the total population may get extinct at finite time, since $\ln |\overline{N}_t|$ is not even defined. To cope with this problem, we first characterize, in a general setting, the event on which an exponential martingale vanishes in the limit (Proposition 2.1.2). We then apply this characterization to the martingale $|\overline{N}_t|$. See also [13] for the application of this idea to the continuous-time setting.

Next, we present a result which says that, under a mild assumption, we can replace

$$\sum_{t\geq 0}\mathcal{R}_t=\infty$$

in (1.24) by a stronger localization property:

$$\overline{\lim_{t\to\infty}}\,\mathcal{R}_t\geq c,$$

where $c>0$ is a constant. To state the theorem, we introduce some notation related to the random walk associated to our model. Let $((S_t)_{t\in\mathbb{N}},P_S^x)$ be the random walk on $\mathbb{Z}^d$ such that:

$$P_S^x(S_0=x)=1 \quad \text{and} \quad P_S^x(S_1=y)=a_{y-x}/|a| \tag{1.28}$$

and let $(\widetilde{S}_t)_{t\in\mathbb{N}}$ be its independent copy. We then define:

$$\pi_d=P_S^0\otimes P_{\widetilde{S}}^0(S_t=\widetilde{S}_t \text{ for some } t\geq 1). \tag{1.29}$$

Then, by (1.11),

$$\pi_d=1 \quad \text{for } d=1,2 \quad \text{and} \quad \pi_d<1 \quad \text{for } d\geq 3. \tag{1.30}$$

Theorem 1.4.2 *Suppose* (1.22) *and either of*

(a) $d=1,2$,
(b) $P(|\overline{N}_\infty|=0)=1$ *and*

$$\gamma>\frac{1}{\pi_d}, \tag{1.31}$$

where γ *and* π_d *are from* (1.22) *and* (1.29).

Then, there exists a constant $c>0$ *such that*:

$$\{\text{survival}\}=\left\{\overline{\lim_{t\to\infty}}\,\mathcal{R}_t\geq c\right\} \quad \textit{a.s.} \tag{1.32}$$

This result generalizes [4, Theorem 1.2] and [5, Proposition 1.4(b)], which are obtained in the context of DPRE. Similar results are also known for branching random walk in random environment [10, Theorem 1.3.2]. To prove Theorem 1.4.2, we will use the argument which was initially applied to DPRE by P. Carmona and Y. Hu in [4] (see also [10]). What is new in the present paper is to carry the arguments in the above mentioned papers over to the case where the extinction at finite time is possible. This will be done in Sect. 3.1.

Remarks (1) We prove (1.32) by way of the following stronger estimate:

$$\underline{\lim}_{t\nearrow\infty}\frac{\sum_{s=0}^t\mathcal{R}_s^{3/2}}{\sum_{s=0}^t\mathcal{R}_s}\geq c_1, \quad \text{a.s.}$$

for some constant $c_1>0$. This in particular implies the following quantitative lower bound on the number of times at which the replica overlap is larger than a certain positive number:

$$\underline{\lim}_{t\nearrow\infty}\frac{\sum_{s=0}^t 1_{\{\mathcal{R}_s\geq c_2\}}}{\sum_{s=0}^t\mathcal{R}_s}\geq c_3, \quad \text{a.s.}$$

where c_2 and c_3 are positive constants. (The inequality $r^{3/2} \le \mathbf{1}\{r \ge c\} + \sqrt{cr}$ for $r, c \in [0, 1]$ can be used here.)

(2) (1.32) is in contrast with the following delocalization result by M. Nakashima [14]: if $d \ge 3$ and $\sup_{t \ge 0} P[|\overline{N}_t|^2] < \infty$, then,

$$\mathcal{R}_t = \mathcal{O}(t^{-d/2}) \quad \text{in } P(\cdot||\overline{N}_\infty| > 0)\text{-probability.}$$

See also [12] for the continuous-time case and [15, 17] for the case of branching random walk in random environment.

Finally, we state the following variant of Theorem 1.4.2, which says that even for $d \ge 3$, (1.31) can be dropped at the cost of some alternative assumptions. Following M. Birkner [2, p. 81, (5.1)], we introduce the following condition:

$$\sup_{t \in \mathbb{N}, x \in \mathbb{Z}^d} \frac{P_S^0(S_t = x)}{P_S^0 \otimes P_{\widetilde{S}}^0(S_t = \widetilde{S}_t)} < \infty, \tag{1.33}$$

which is obviously true for the symmetric simple random walk on $\mathbb{Z}^d$.

Theorem 1.4.3 *Suppose $d \ge 3$, (1.22), (1.33) and that there exist mean-one i.i.d. random variables $\overline{\eta}_{t,y}$, $(t, y) \in \mathbb{N} \times \mathbb{Z}^d$ such that*:

$$A_{t,x,y} = \overline{\eta}_{t,y} a_{y-x}. \tag{1.34}$$

Then, the slow growth ($P(|N_\infty| = 0) = 1$) implies that there exists a constant $c > 0$ such that (1.32) holds.

Note that OSP and DPRE for $d \ge 3$ satisfy all the assumptions for Theorem 1.4.3. The proof of Theorem 1.4.3 is based on Theorem 1.4.2 and a criterion for the regular growth phase, which is essentially due to M. Birkner [3]. These will be explained in Sect. 3.4.

Proof of Theorem 1.1.1 The theorem follows from Theorems 1.4.1 and 1.4.3. □

2 Proofs of Theorem 1.4.1

We will prove part (b) first, and then part (a).

2.1 An Abstraction of Theorem 1.4.1(b)

We will prove Theorem 1.4.1(b) in the following generalized form, where the slow glowth ($P(|\overline{N}_\infty| = 0) = 1$) is not assumed in advance:

Proposition 2.1.1

(a) *Even without assuming (1.22) and with (1.8) replaced by (1.26), it holds that*

$$\{|\overline{N}_\infty| > 0\} \supset \left\{ \text{survival}, \sum_{t \ge 0} \mathcal{R}_t < \infty \right\} \quad a.s. \tag{2.1}$$

(b) *Suppose* (1.8) *and* (1.22). *Then, there exists a constant* $c > 0$ *such that* (1.25) *holds a.s. on the event* $\{\sum_{t\geq 0} \mathcal{R}_t = \infty\}$. *In particular, the inclusion opposite to* (2.1) *holds true.*

We will prove Proposition 2.1.1 via the following observation for general exponential martingales, which may be of independent interest.

Let $(M_t)_{t\in\mathbb{N}}$ be a square-integrable martingale on a filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in\mathbb{N}})$. We denote its predictable quadratic variation by:

$$\langle M\rangle_t = \sum_{1\leq u\leq t} P[(\Delta M_u)^2|\mathcal{F}_{u-1}].$$

Here, and in what follows, we write $\Delta a_t = a_t - a_{t-1}$ $(t \geq 1)$ for a sequence $(a_t)_{t\in\mathbb{N}}$ (random or non-random).

Proposition 2.1.2 *Let* $(Y_t)_{t\in\mathbb{N}}$ *be a mean-zero square-integrable martingale on a filtered probability space* $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t\in\mathbb{N}})$ *such that* $-1 \leq \Delta Y_t$ *a.s. for all* $t \in \mathbb{N}^*$ *and let*

$$X_t = \prod_{s=1}^{t}(1 + \Delta Y_s). \tag{2.2}$$

(a) *Suppose that*

$$\sup_{t\geq 1} P[(\Delta Y_t)^2|\mathcal{F}_{t-1}] \leq c_1^2 \quad a.s. \tag{2.3}$$

for some constant $c_1 \in (0,\infty)$. *Then,*

$$\{X_\infty > 0\} \supset S \cap \{\langle Y\rangle_\infty < \infty\} \quad a.s. \tag{2.4}$$

where $S = \{X_t > 0 \text{ for all } t \geq 0\}$.

(b) *Suppose that there exists a constant* $c_2 \in (0,\infty)$ *such that for all* $t \in \mathbb{N}^*$:

$$Y_t \in L^3(P) \quad and \quad P[(\Delta Y_t)^3|\mathcal{F}_{t-1}] \leq c_2 P[(\Delta Y_t)^2|\mathcal{F}_{t-1}] \quad a.s. \tag{2.5}$$

Then, for any $c_3 \in (0, \frac{1}{4})$,

$$X_t \leq \exp(-c_3\langle Y\rangle_t) \quad \text{for all large enough } t\text{'s} \tag{2.6}$$

a.s. on the event $\{\langle Y\rangle_\infty = \infty\}$. *In particular, the inclusion opposite to* (2.4) *holds true.*

Remark As will be seen from the proof, the following assumption works as well for Proposition 2.1.2(b): there exist $q \in (2,\infty)$ and $c_2 \in (0,\infty)$ such that for all $t \in \mathbb{N}^*$:

$$Y_t \in L^q(P) \quad \text{and} \quad P[|\Delta Y_t|^q|\mathcal{F}_{t-1}] \leq c_2^{q-2} P[(\Delta Y_t)^2|\mathcal{F}_{t-1}] \quad \text{a.s.}$$

Although this condition may look better than (2.5) for $q < 3$, (2.5) works more effectively for our application. The point is that (2.5) is written in terms of $(\Delta Y_t)^3$, rather than $|\Delta Y_t|^3$.

We postpone the proof of Proposition 2.1.2 (Sect. 2.2) to finish the proof of Proposition 2.1.1.

Proof of Proposition 2.1.1 We apply Proposition 2.1.2 to $X_t = |\overline{N}_t|$. Then, it is easy to see that (2.2) holds with:

$$\Delta Y_t = \frac{1}{|a|} \sum_{x,y \in \mathbb{Z}^d} \rho_{t-1}(x) A_{t,x,y} - 1.$$

Moreover, it was shown in the proof of [18, Lemma 3.2.2] that there are constants $c_i \in (0, \infty)$ $(i = 1, 2)$ such that:

$$(1) \quad P[(\Delta Y_t)^p | \mathcal{F}_{t-1}] \le c_1 \mathcal{R}_{t-1}, \quad p = 2, 3$$

$$(2) \quad P[(\Delta Y_t)^2 | \mathcal{F}_{t-1}] \ge c_2 \mathcal{R}_{t-1}$$

((1.22) is used only for (2)). Therefore, Proposition 2.1.2 immediately leads to Theorem 1.4.1. □

2.2 Proof of Proposition 2.1.2

Let $(M_t)_{t \in \mathbb{N}}$ be a square-integrable martingale defined on a filtered probability space. In this paper, we will repeatedly exploit the following well-known facts (e.g., [8, pp. 252–253]):

$$\{\langle M \rangle_\infty < \infty\} \subset \{M_t \text{ converges as } t \to \infty\} \quad \text{a.s.,} \tag{2.7}$$

$$\{\langle M \rangle_\infty = \infty\} \subset \left\{ \lim_{t \to \infty} \frac{M_t}{\langle M \rangle_t} = 0 \right\} \quad \text{a.s.} \tag{2.8}$$

To prove Proposition 2.1.2, we will use the following lemma, which is a generalization of the Borel–Cantelli lemma, and is also used in the proof of Lemma 3.1.2.

Lemma 2.2.1 *Let $(Z_t)_{t \in \mathbb{N}}$ be an integrable, adapted process defined on a filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t)_{t \in \mathbb{N}})$ and let*:

$$A_0 = 0, \qquad A_t = \sum_{1 \le s \le t} P[\Delta Z_s | \mathcal{F}_{s-1}], \quad t \in \mathbb{N}^*.$$

(a) *Suppose that there exists a constant $c_1 \in (0, \infty)$ such that*:

$$\Delta Z_t - P[\Delta Z_t | \mathcal{F}_{t-1}] \ge -c_1 \quad \textit{a.s. for all } t \in \mathbb{N}^*. \tag{2.9}$$

Then,

$$\left\{ \lim_{t \to \infty} Z_t = \infty \right\} = \left\{ \lim_{t \to \infty} Z_t = \infty,\ \varlimsup_{t \to \infty} \frac{A_t}{Z_t} \ge 1 \right\} \subset \left\{ \sup_{t \ge 1} A_t = \infty \right\} \quad \textit{a.s.} \tag{2.10}$$

(b) *Suppose that $\{Z_t\}_{t \in \mathbb{N}} \subset L^2(P)$ and that there exists a constant $c_2 \in (0, \infty)$ such that*:

$$\operatorname{var}(\Delta Z_t | \mathcal{F}_{t-1}) \le c_2 P[\Delta Z_t | \mathcal{F}_{t-1}] \quad \textit{a.s. for all } t \in \mathbb{N}^*, \tag{2.11}$$

where $\operatorname{var}(\Delta Z_t | \mathcal{F}_{t-1}) = P[(\Delta Z_t)^2 | \mathcal{F}_{t-1}] - P[\Delta Z_t | \mathcal{F}_{t-1}]^2$. *Then*,

$$\left\{ \lim_{t \to \infty} A_t = \infty \right\} = \left\{ \lim_{t \to \infty} A_t = \infty,\ \lim_{t \to \infty} \frac{Z_t}{A_t} = 1 \right\} \subset \left\{ \lim_{t \to \infty} Z_t = \infty \right\} \quad \textit{a.s.} \tag{2.12}$$

Proof (a) It is enough to show that

$$(1)\quad \left\{\lim_{t\to\infty} Z_t = \infty\right\} \subset \left\{\overline{\lim_{t\to\infty}} \frac{A_t}{Z_t} \geq 1\right\}.$$

Define $M_t = Z_t - A_t$, so that $(M_\cdot)$ is a martingale whose increments are bounded below by $-c_1$. Then, it is standard (e.g. the proof of [8, p. 236, (3.1)]) that

$$(2)\quad P(C \cup D_-) = 1,$$

where

$$C = \{M_t \text{ converges as } t \to \infty\} \quad \text{and} \quad D_- = \left\{\inf_{t\in\mathbb{N}} M_t = -\infty\right\}.$$

Now, by writing

$$\frac{A_t}{Z_t} = 1 - \frac{M_t}{Z_t},$$

(1) follows immediately from (2).

(b) It is enough to show that

$$(3)\quad \left\{\lim_{t\to\infty} A_t = \infty\right\} \subset \left\{\lim_{t\to\infty} \frac{Z_t}{A_t} = 1\right\}.$$

Here, $M_\cdot$ is square-integrable. Since

$$\left|\frac{Z_t}{A_t} - 1\right| = \left|\frac{M_t}{A_t}\right|,$$

we have

$$\left\{\lim_{t\to\infty} A_t = \infty,\ \langle M\rangle_\infty < \infty\right\} \overset{(2.7)}{\subset} \left\{\lim_{t\to\infty} \frac{Z_t}{A_t} = 1\right\}.$$

On the other hand, on the event $\{\langle M\rangle_\infty = \infty\}$, we have

$$\left|\frac{M_t}{A_t}\right| \overset{(2.11)}{\leq} c_2 \frac{|M_t|}{\langle M\rangle_t} \overset{(2.8)}{\longrightarrow} 0 \quad \text{as } t \to \infty.$$

These prove (3). □

Remark Similarly as Lemma 2.2.1(a), we can show the following variant of Lemma 2.2.1(b). Suppose that there exists a constant $c_3 \in (0, \infty)$ such that:

$$\Delta Z_t - P[\Delta Z_t | \mathcal{F}_{t-1}] \leq c_3 \quad \text{a.s. for all } t \in \mathbb{N}^*.$$

Then,

$$\left\{\lim_{t\to\infty} A_t = \infty\right\} = \left\{\lim_{t\to\infty} A_t = \infty,\ \overline{\lim_{t\to\infty}} \frac{Z_t}{A_t} \geq 1\right\} \subset \left\{\sup_{t\geq 1} Z_t = \infty\right\} \quad \text{a.s.}$$

Lemma 2.2.2 *Let* $(Y_t)_{t\in\mathbb{N}^*}$ *be as in Proposition* 2.1.2(b). *Then,*

$$\{\langle Y\rangle_\infty=\infty\}\subset\left\{\varliminf_{t\to\infty}\frac{\sum_{s\le t}f(\Delta Y_s)}{\langle Y\rangle_t}\ge 1\right\}\quad a.s. \tag{2.13}$$

where $f(u)=\frac{u^2}{2+u}$, $u\ge -1$.

Proof We first prepare elementary estimates. Let U be a r.v. such that $-1\le U$ a.s. Since $0\le f(u)\vee f(u)^2\le u^2$, we have

$$(1)\quad P\left[f(U)\vee f(U)^2\right]\le P[U^2].$$

Suppose further that $P[U^3]\le cP[U^2]$. Then,

$$(2)\quad P[U^2]\vee P[f(U)^2]\le (2+c)P[f(U)].$$

This can be seen as follows. We have

$$\begin{aligned}P[U^2]^2&=P\left[\frac{U}{\sqrt{2+U}}U\sqrt{2+U}\right]^2\le P\left[f(U)\right]P\left[U^2(2+U)\right]\\&=P\left[f(U)\right](2P[U^2]+P[U^3])\le(2+c)P[f(U)]P[U^2],\end{aligned}$$

which proves $P[U^2]\le(2+c)P[f(U)]$. On the other hand,

$$P[f(U)^2]\overset{(1)}{\le}P[U^2]\le(2+c)P[f(U)].$$

By (1)–(2) above, applied to $U=\Delta Y_t$ and the measure $P(\,\cdot\,|\mathcal{F}_{t-1})$, we see that

$$(3)\quad D\overset{\text{def}}{=}\{\langle Y\rangle_\infty=\infty\}=\left\{\sum_{s\ge1}P[f(\Delta Y_s)|\mathcal{F}_{s-1}]=\infty\right\}\quad\text{a.s.}$$

We see from (2) that $Z_t=\sum_{s\le t}f(\Delta Y_s)$ satisfies (2.11). Therefore,

$$D\overset{(3),(2.12)}{\subset}\left\{\lim_{t\to\infty}\frac{\sum_{s\le t}f(\Delta Y_s)}{\sum_{1\le s\le t}P[f(\Delta Y_s)|\mathcal{F}_{s-1}]}=1\right\}\quad\text{a.s.}$$

Thus, (2.13) follows from this and (1). □

Proof of Proposition 2.1.2(a) We will prove that

$$(1)\quad S\cap\{\langle Y\rangle_\infty<\infty\}\subset\{\exp(-Y_\infty)X_\infty>0\}\quad\text{a.s.}$$

We get (2.4) from this and (2.7). To prove (1), note that

$$(2)\quad \exp(-Y_t)X_t=\prod_{u=1}^{t}(1+\Delta Y_u)\exp(-\Delta Y_u)$$

and that

$$(3)\quad 0\le 1-(1+\Delta Y_u)\exp(-\Delta Y_u)\le\frac{e}{2}(\Delta Y_u)^2,$$

since $\Delta Y_u \geq -1$. By (2.3), $Z_t = \sum_{s \leq t} (\Delta Y_s)^2$ satisfies (2.9). Thus, we have by (2.10) that

$$(4) \quad \{\langle Y \rangle_\infty < \infty\} \subset \left\{ \sum_{u \geq 1} (\Delta Y_u)^2 < \infty \right\} \quad \text{a.s.}$$

Thus, we get (1) from (2)–(4).

(b) We have $(1+u)e^{-u} \leq e^{-f(u)/4}$ for $u \geq -1$, where $f(u) = \frac{u^2}{2+u}$. Thus,

$$(5) \quad (1 + \Delta Y_u) \exp(-\Delta Y_u) \leq \exp(-f(\Delta Y_u)/4) \quad \text{for all } u \geq 1.$$

Let $0 < c_3 < c_4 < \frac{1}{4}$. Then, for t large enough, a.s. on the event $\{\langle Y \rangle_\infty = \infty\}$,

$$\begin{aligned} \prod_{u=1}^{t} (1 + \Delta Y_u) \exp(-\Delta Y_u) &\overset{(5)}{\leq} \exp\left(-\sum_{u=1}^{t} f(\Delta Y_u)/4 \right) \overset{(2.13)}{\leq} \exp\left(-c_4 \langle Y \rangle_t\right) \\ &\overset{(2.8)}{\leq} \exp\left(-Y_t - c_3 \langle Y \rangle_t\right), \end{aligned}$$

which, via (2), proves (2.6). □

2.3 Proof of Theorem 1.4.1(a)

If $P(|\overline{N}_\infty| > 0) > 0$, then,

$$\{\text{survival}\} = \{|\overline{N}_\infty| > 0\} \quad \text{a.s.}$$

This can be seen easily by translating the argument in [9, p. 701, proof of "Proposition"]. We see from this and Proposition 2.1.1 that $\sum_{t \geq 0} \mathcal{R}_t < \infty$ a.s. on the event of survival, while $\sum_{t \geq 0} \mathcal{R}_t < \infty$ is obvious outside the event of survival.

3 Proofs of Theorems 1.4.2 and 1.4.3

3.1 The Argument by P. Carmona and Y. Hu

For $f, g : \mathbb{Z}^d \to [0, \infty)$, we define their convolution $f * g$ by:

$$(f * g)(x) = \sum_{y \in \mathbb{Z}^d} f(x-y) g(y), \quad x \in \mathbb{Z}^d.$$

For the notational convenience, we also write $a(y)$ for a_y. We define:

$$b_t = \underbrace{b * \cdots * b}_{t}, \quad t \in \mathbb{N}^*, \quad \text{where } b(x) = \frac{1}{|a|^2} \sum_{y \in \mathbb{Z}^d} a(y) a(y-x).$$

To interpret this, let $(\widetilde{S}_t)_{t \in \mathbb{N}}$ be the independent copy of $((S_t)_{t \in \mathbb{N}}, P_S^0)$, cf. (1.28). Then,

$$b_t(x) = P_S^0 \otimes P_{\widetilde{S}}^0 (S_t - \widetilde{S}_t = x).$$

Therefore, by (1.11)

$$1+\sum_{t\geq 1} b_t(0)=\frac{1}{1-\pi_d}\begin{cases}=\infty & \text{if } d=1,2\\ <\infty & \text{if } d\geq 3.\end{cases} \tag{3.1}$$

We first note that there are $\varepsilon>0$ and $t_0\in\mathbb{N}$ such that:

$$\sum_{1\leq t\leq t_0} b_t(0)\geq \frac{1+\varepsilon}{\gamma-1}. \tag{3.2}$$

For $d=1,2$, we take $\varepsilon=1$. Then, (3.2) holds for t_0 large enough, since $\sum_{t\geq 1} b_t(0)=\infty$. For $d\geq 3$, the assumptions (1.31) and (3.1) imply (3.2) for small enough $\varepsilon>0$ and large enough t_0. We now fix $\varepsilon>0$ and t_0 and define:

$$X_t=\langle g*\rho_t,\rho_t\rangle,\quad \text{where } g=\sum_{s=1}^{t_0} b_s. \tag{3.3}$$

(The bracket $\langle\cdot,\,\cdot\rangle$ stands for the inner product of $\ell^2(\mathbb{Z}^d)$.) Note that $0\leq g\in\ell^1(\mathbb{Z}^d)$ and that

$$|X_t|\leq |(g*\rho_t)^2|^{1/2}|\rho_t^2|^{1/2}\leq |g|\mathcal{R}_t. \tag{3.4}$$

(Recall again that $|f|=\sum_x |f(x)|$ for $f:\mathbb{Z}^d\to\mathbb{R}$.) Let:

$$X_t=M_t+A_t$$

be Doob's decomposition, defined by:

$$A_0=0,\qquad \Delta A_t=P[\Delta X_t|\mathcal{F}_{t-1}]\quad \text{for } t\in\mathbb{N}^*. \tag{3.5}$$

Proof of Theorem 1.4.2 is based on the following two lemmas.

Lemma 3.1.1 *There are constants $c_1,c_2\in(0,\infty)$ such that*:

$$A_t\geq c_1\sum_{0\leq u\leq t-1}\mathcal{R}_u-c_2\sum_{0\leq u\leq t-1}\mathcal{R}_u^{3/2}\quad \textit{for all } t\in\mathbb{N}^*.$$

Lemma 3.1.2

$$\left\{\sum_{u\geq 0}\mathcal{R}_u=\infty\right\}\subset\left\{\lim_{t\to\infty}\frac{M_t}{\sum_{0\leq u\leq t-1}\mathcal{R}_u}=0\right\}\quad a.s.$$

Proof of Theorem 1.4.2 We may focus on the event $D=\{\sum_{u\geq 0}\mathcal{R}_u=\infty\}$. It follows from (3.4) and Lemma 3.1.2 that

$$\lim_{t\to\infty}\frac{A_t}{\sum_{0\leq u\leq t-1}\mathcal{R}_u}=0\quad \text{a.s. on } D$$

and hence from Lemma 3.1.1 that

$$\varliminf_{t\to\infty}\frac{\sum_{0\leq u\leq t-1}\mathcal{R}_u^{3/2}}{\sum_{0\leq u\leq t-1}\mathcal{R}_u}\geq\frac{c_1}{c_2}\quad \text{a.s. on } D.$$

This, together with (1.24), proves Theorem 1.4.2. □

3.2 Proof of Lemma 3.1.1

The following technical lemma is an extension of [10, Lemma 3.1.1] to the case where the random variables $U_i \geq 0$ may vanish with positive probability.

Lemma 3.2.1 *Let $U_i \geq 0$, $1 \leq i \leq n$ $(n \geq 2)$ be independent random variables such that*:

$$P[U_i^3] < \infty \quad \text{for } i = 1, \dots, n \text{ and } \sum_{i=1}^n m_i = 1,$$

where $m_i = P[U_i]$. Then, with $U = \sum_{i=1}^n U_i$,

$$P\left[\frac{U_1 U_2}{U^2} : U > 0\right] \geq m_1 m_2 - 2m_2 \operatorname{var}(U_1) - 2m_1 \operatorname{var}(U_2), \tag{3.6}$$

$$P\left[\frac{U_1^2}{U^2} : U > 0\right] \geq P[U_1^2](1 + 2m_1) - 2P[U_1^3]. \tag{3.7}$$

Proof Note that $x^{-2} \geq 3 - 2x$ for $x \in (0, \infty)$. Thus, we have that

$$\begin{aligned} P\left[\frac{U_1 U_2}{U^2} : U > 0\right] &\geq P\left[U_1 U_2 (3 - 2U) : U > 0\right] = P\left[U_1 U_2 (3 - 2U)\right] \\ &= P\left[U_1 U_2 (1 - 2(U - 1))\right] = m_1 m_2 - 2P\left[U_1 U_2 (U - 1)\right], \\ P\left[U_1 U_2 (U - 1)\right] &= P\left[U_1 U_2 (U_1 - m_1)\right] + P\left[U_1 U_2 (U_2 - m_2)\right] \\ &= m_2 \operatorname{var}(U_1) + m_1 \operatorname{var}(U_2). \end{aligned}$$

These prove (3.6). Similarly,

$$\begin{aligned} P\left[\frac{U_1^2}{U^2} : U > 0\right] &\geq P\left[U_1^2 (3 - 2U) : U > 0\right] = P\left[U_1^2 (3 - 2U)\right] \\ &= P\left[U_1^2\right] - 2P\left[U_1^2 (U - 1)\right], \\ P\left[U_1^2 (U - 1)\right] &= P\left[U_1^2 (U_1 - m_1)\right] = P\left[U_1^3\right] - m_1 P\left[U_1^2\right]. \end{aligned}$$

These prove (3.7). □

We introduce

$$\rho_{t,1} = \rho_t * \overline{a}, \qquad \mathcal{R}_{t,1} = |\rho_{t,1}^2|, \tag{3.8}$$

where $\overline{a}(x) = a(x)/|a|$, $x \in \mathbb{Z}^d$.

We will make a series of estimates on quantities involving $a(x)$, $\rho_t(x)$, $\mathcal{R}_t$, and so on. In the sequel, multiplicative constants are denoted by $c, c_1, c_2, \dots$. We agree that they are *non-random* constants which do not depend on time variables $t, s, \dots \in \mathbb{N}$ or space variables $x, y, \dots \in \mathbb{Z}^d$.

Lemma 3.2.2 *For any $t \in \mathbb{N}$,*

$$\mathcal{R}_{t,1} \leq \mathcal{R}_t \leq \frac{|a|^2}{|a^2|} \mathcal{R}_{t,1}. \tag{3.9}$$

Proof Let $\overline{a}(x) = a(x)/|a|$, $x \in \mathbb{Z}^d$. We then have

$$|\rho_{t,1}^2| = |(\rho_t * \overline{a})^2| \le |\rho_t^2|$$

by Young's inequality. This proves the first inequality. On the other hand,

$$\begin{aligned}|\rho_{t,1}^2| = |(\rho_t * \overline{a})^2| &= \sum_{x \in \mathbb{Z}^d} \left(\sum_{y \in \mathbb{Z}^d} \rho_t(x-y)\overline{a}(y) \right)^2 \\ &\ge \sum_{x \in \mathbb{Z}^d} \sum_{y \in \mathbb{Z}^d} \rho_t(x-y)^2 \overline{a}(y)^2 = |\rho_t^2||\overline{a}^2|,\end{aligned}$$

which proves the second inequality. □

We assume (1.22) from here on.

Lemma 3.2.3 *There is a constant* $c \in (0, \infty)$ *such that the following hold*:

$$\begin{aligned}&P\left[\rho_t(y)\rho_t(\widetilde{y})|\mathcal{F}_{t-1}\right] \\ &\quad \ge \rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y}) - c\rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y})^2 - c\rho_{t-1,1}(\widetilde{y})\rho_{t-1,1}(y)^2, \qquad (3.10)\end{aligned}$$

for all $t \in \mathbb{N}^*$, $y, \widetilde{y} \in \mathbb{Z}^d$ *with* $y \ne \widetilde{y}$.

$$P\left[\mathcal{R}_t|\mathcal{F}_{t-1}\right] \ge \gamma \mathcal{R}_{t-1,1} - c\mathcal{R}_{t-1,1}^{3/2} \quad \textit{for all } t \in \mathbb{N}^*. \qquad (3.11)$$

Proof Let $U_t = \sum_{y \in \mathbb{Z}^d} U_{t,y}$, where $U_{t,y} = \frac{1}{|a|}\sum_{x \in \mathbb{Z}^d} \rho_{t-1}(x) A_{t,x,y}$. Then, $\{U_{t,y}\}_{y \in \mathbb{Z}^d}$ are independent under $P(\cdot|\mathcal{F}_{t-1})$. Moreover, it is not difficult to see that (cf. proof of [18, Lemma 3.2.2]), on the event $\{|\overline{N}_{t-1}| > 0\}$,

$$(1) \quad P[U_{t,y}|\mathcal{F}_{t-1}] = \rho_{t-1,1}(y), \qquad P[U_t|\mathcal{F}_{t-1}] = 1,$$

$$(2) \quad P[U_{t,y}^2|\mathcal{F}_{t-1}] = \frac{1}{|a|^2} \sum_{x_1,x_2,y \in \mathbb{Z}^d} \rho_{t-1}(x_1)\rho_{t-1}(x_2) P[A_{t,x_1,y}A_{t,x_2,y}],$$

$$(3) \quad P[U_{t,y}^m|\mathcal{F}_{t-1}] \le c_1 \rho_{t,1}(y)^m, \quad m = 2, 3.$$

Since

$$\rho_t(y)\rho_t(\widetilde{y}) = (U_{t,y}U_{t,\widetilde{y}}/U_t)\mathbf{1}_{\{|\overline{N}_{t-1}|>0\}}$$

and $\{U_t > 0\} \subset \{|\overline{N}_{t-1}| > 0\}$, we see from (1), (3) above and Lemma 3.2.1 that (3.10) holds and that

$$(4) \quad P\left[\rho_t(y)^2|\mathcal{F}_{t-1}\right] \ge P[U_{t,y}^2|\mathcal{F}_{t-1}] - 2c_1\rho_{t-1,1}(y)^3.$$

To prove (3.11), note that

$$(5) \quad \sum_{y \in \mathbb{Z}^d} \rho_{t-1,1}(y)^3 \le \left(\sum_{y \in \mathbb{Z}^d} \rho_{t-1,1}(y)^2 \right)^{3/2} = \mathcal{R}_{t-1,1}^{3/2}.$$

We then see that

$$
\begin{aligned}
P\left[\mathcal{R}_t|\mathcal{F}_{t-1}\right] &\overset{(4)}{\geq} \sum_{y\in\mathbb{Z}^d}\left(P[U_{t,y}^2|\mathcal{F}_{t-1}] - 2c_1\rho_{t-1,1}(y)^3\right)\\
&\overset{(2),(5)}{\geq} \frac{1}{|a|^2}\sum_{x_1,x_2,y\in\mathbb{Z}^d}\rho_{t-1}(x_1)\rho_{t-1}(x_2)P[A_{t,x_1,y}A_{t,x_2,y}] - 2c_1\mathcal{R}_{t-1,1}^{3/2}\\
&\overset{(1.22)}{\geq} \frac{\gamma}{|a|^2}\sum_{x_1,x_2,y\in\mathbb{Z}^d}\rho_{t-1}(x_1)\rho_{t-1}(x_2)a(y-x_1)a(y-x_2) - 2c_1\mathcal{R}_{t-1,1}^{3/2}\\
&= \gamma\mathcal{R}_{t-1,1} - 2c_1\mathcal{R}_{t-1,1}^{3/2}.
\end{aligned}
$$

□

Proof of Lemma 3.1.1

$$
P[X_t|\mathcal{F}_{t-1}] = \sum_{y,\widetilde{y}\in\mathbb{Z}^d} g(y-\widetilde{y})P[\rho_t(y)\rho_t(\widetilde{y})|\mathcal{F}_{t-1}] = I + J,
$$

where I and J are diagonal and off-diagonal terms:

$$
\begin{aligned}
I &= g(0)\sum_{y\in\mathbb{Z}^d} P[\rho_t(y)^2|\mathcal{F}_{t-1}],\\
J &= \sum_{\substack{y,\widetilde{y}\in\mathbb{Z}^d\\ y\neq\widetilde{y}}} g(y-\widetilde{y})P[\rho_t(y)\rho_t(\widetilde{y})|\mathcal{F}_{t-1}].
\end{aligned}
$$

We start with the lower bound for I.

$$
(1)\quad I = g(0)P[\mathcal{R}_t|\mathcal{F}_{t-1}] \overset{(3.11)}{\geq} g(0)\gamma\mathcal{R}_{t-1,1} - g(0)c\mathcal{R}_{t-1,1}^{3/2}.
$$

As for J, we have

$$
J \overset{(3.10)}{\geq} J_{1,1} - cJ_{1,2} - cJ_{2,1},
$$

where

$$
J_{m,n} = \sum_{\substack{y,\widetilde{y}\in\mathbb{Z}^d\\ y\neq\widetilde{y}}} g(y-\widetilde{y})\rho_{t-1,1}(y)^m\rho_{t-1,1}(\widetilde{y})^n.
$$

$J_{1,1}$ can be computed exactly:

$$
\begin{aligned}
(2)\quad J_{1,1} &= \left(\sum_{y,\widetilde{y}\in\mathbb{Z}^d} - \sum_{\substack{y,\widetilde{y}\in\mathbb{Z}^d\\ y=\widetilde{y}}}\right) g(y-\widetilde{y})\rho_{t-1,1}(y)\rho_{t-1,1}(\widetilde{y})\\
&= \langle g * b * \rho_{t-1}, \rho_{t-1}\rangle - g(0)\mathcal{R}_{t-1,1}.
\end{aligned}
$$

To bound $J_{1,2}$ from above, note that

$$
\max_{x\in\mathbb{Z}^d}(g*\rho_{t-1,1})(x) \leq |g|\max_{x\in\mathbb{Z}^d}\rho_{t-1,1}(x) \leq |g|\mathcal{R}_{t-1,1}^{1/2}.
$$

Thus,

$$J_{1,2} \le \langle g * \rho_{t-1,1}, \rho^2_{t-1,1} \rangle \le \max_{x \in \mathbb{Z}^d} (g * \rho_{t-1,1})(x) \mathcal{R}_{1,t-1} \le |g| \mathcal{R}^{3/2}_{t-1,1}.$$

Similarly, $J_{2,1} \le |g|\mathcal{R}^{3/2}_{t-1,1}$. Putting things together, we see that

$$\begin{aligned} (3) \quad \Delta A_t &= P[X_t|\mathcal{F}_{t-1}] - X_{t-1} \ge I + J_{1,1} - X_{t-1} - 2c|g|\mathcal{R}^{3/2}_{t-1,1} \\ &\overset{(1)\text{–}(2)}{\ge} (\gamma - 1) g(0) \mathcal{R}_{t-1,1} + \langle (g * b - g) * \rho_{t-1}, \rho_{t-1} \rangle - 3|g|c\mathcal{R}^{3/2}_{t-1,1}. \end{aligned}$$

Note that $g * b - g = b_{t_0+1} - b \ge -b$ and hence that

$$(4) \quad \langle (g * b - g) * \rho_{t-1}, \rho_{t-1} \rangle \ge -\langle b * \rho_{t-1}, \rho_{t-1} \rangle = -\mathcal{R}_{t-1,1}.$$

Therefore,

$$\Delta A_t \overset{(3)\text{–}(4)}{\ge} ((\gamma - 1) g(0) - 1) \mathcal{R}_{t-1,1} - 3|g|c\mathcal{R}^{3/2}_{t-1,1} \overset{(3.2)}{\ge} \varepsilon \mathcal{R}_{t-1,1} - 3|g|c\mathcal{R}^{3/2}_{t-1,1}.$$

We now get Lemma 3.1.1 from this and (3.9). □

3.3 Proof of Lemma 3.1.2

We have

$$\left\{ \sum_{1 \le u < \infty} \mathcal{R}_u = \infty, \ \langle M \rangle_\infty < \infty \right\} \overset{(2.7)}{\subset} \left\{ \lim_{t \to \infty} \frac{M_t}{\sum_{1 \le u \le t} \mathcal{R}_u} = 0 \right\} \quad \text{a.s.}$$

To treat the case of $\langle M \rangle_\infty = \infty$, we show that

$$(1) \quad \langle M \rangle_t \le 4|g|^2 \sum_{1 \le u \le t} (\mathcal{R}_{u-1} + P[\mathcal{R}_u|\mathcal{F}_{u-1}]).$$

We have

$$(2) \quad |\Delta X_t|^2 \overset{(3.4)}{\le} 2|g|^2 (\mathcal{R}^2_t + \mathcal{R}^2_{t-1}) \le 2|g|^2 (\mathcal{R}_t + \mathcal{R}_{t-1}),$$

and

$$(3) \quad (\Delta A_t)^2 \overset{\text{Schwarz}}{\le} P[(\Delta X_t)^2|\mathcal{F}_{t-1}] \overset{(2)}{\le} 2|g|^2 (P[\mathcal{R}_t|\mathcal{F}_{t-1}] + \mathcal{R}_{t-1}).$$

Thus,

$$\begin{aligned} \Delta \langle M \rangle_t &= P[(\Delta M_t)^2|\mathcal{F}_{t-1}] \le 2P[(\Delta X_t)^2|\mathcal{F}_{t-1}] + 2(\Delta A_t)^2 \\ &\overset{(3)}{\le} 4|g|^2 (P[\mathcal{R}_t|\mathcal{F}_{t-1}] + \mathcal{R}_{t-1}). \end{aligned}$$

Now, we have by Lemma 2.2.1 and (1) that

$$\begin{aligned} \left\{ \sum_{1 \le u < \infty} \mathcal{R}_u = \infty \right\} &\overset{(2.10)}{=} \left\{ \sum_{1 \le u < \infty} P[\mathcal{R}_u|\mathcal{F}_{u-1}] = \infty \right\} \quad \text{a.s.} \\ &\overset{(2.12)}{=} \left\{ \lim_{t \to \infty} \frac{\sum_{1 \le u \le t} \mathcal{R}_u}{\sum_{1 \le u \le t} P[\mathcal{R}_u|\mathcal{F}_{u-1}]} = 1 \right\} \quad \text{a.s.} \end{aligned}$$

$$\stackrel{(1)}{\subset} \left\{ \varliminf_{t\to\infty} \frac{\sum_{1\le u\le t} \mathcal{R}_u}{\langle M\rangle_t} \geq \frac{1}{4|g|^2} \right\}.$$

We see from this and (2.8) that

$$\left\{ \sum_{1\le u<\infty} \mathcal{R}_u = \infty,\ \langle M\rangle_\infty = \infty \right\} \subset \left\{ \lim_{t\to\infty} \frac{M_t}{\sum_{1\le u\le t} \mathcal{R}_u} = 0 \right\} \quad \text{a.s.}$$

This completes the proof of Lemma 3.1.2.

3.4 Proof of Theorem 1.4.3

We now state a criterion for the regular growth phase (Lemma 3.4.1). The criterion is an extension of the one obtained by M. Birkner [3] for DPRE.

Let $((S_t)_{t\in\mathbb{N}}, P_S^x)$ be the random walk defined by (1.28) and let $(\widetilde{S}_t)_{t\in\mathbb{N}}$ be its independent copy. Since the random variable:

$$V_\infty(S,\widetilde{S}) = \sum_{t\ge 1} \mathbf{1}_{\{S_t=\widetilde{S}_t\}}$$

is geometrically distributed with the parameter π_d, we have

$$\frac{1}{\pi_d} = \sup\left\{\alpha \geq 1;\ P_S^0 \otimes P_{\widetilde{S}}^0\left[\alpha^{V_\infty(S,\widetilde{S})}\right] < \infty\right\}. \tag{3.12}$$

We now define π_d^* by:

$$\frac{1}{\pi_d^*} = \sup\left\{\alpha \geq 1;\ P_{\widetilde{S}}^0\left[\alpha^{V_\infty(S,\widetilde{S})}\right] < \infty\ P_S^0\text{-a.s.}\right\}. \tag{3.13}$$

Therefore, $\pi_d^* \le \pi_d$ in general. Moreover, the inequality is known to be strict if $d \geq 3$ and (1.33) is satisfied [2, p. 82, Corollary 4].

Lemma 3.4.1 *Suppose $d \geq 3$ and* (1.34). *Then,*

$$P[\overline{\eta}_{t,y}^2] < \frac{1}{\pi_d^*} \quad \Rightarrow \quad P[|\overline{N}_\infty|] = 1.$$

Proof Because of (1.34), we have that

$$N_{t,x} = |a|^t P_S^0\left[\prod_{u=1}^t \overline{\eta}_{u,S_u}\right].$$

Using this expression, we can repeat the argument in [3] without change. (Here, unlike the DPRE case, we may have $P(\overline{\eta}_{t,y} = 0) > 0$. However, this does not cause any problem as far as to prove this lemma.) □

Proof of Theorem 1.4.3 (1.32) $\subset$: Note that $\pi_d^* < \pi_d$ if $d \geq 3$ and (1.33) is satisfied. If $|\overline{N}_\infty| = 0$ a.s., then we have by Lemma 3.4.1 that $\gamma \geq \frac{1}{\pi_d^*} > \frac{1}{\pi_d}$. Thus, we can apply Theorems 1.4.1 and 1.4.2.

(1.32) $\supset$: Obvious. □

Acknowledgements The author thanks Yukio Nagahata for discussions which lead to the simplification of the proof of Theorem 1.4.2.

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